DYNAMIC SPATIAL MODELLING:

DIFFERENTIAL EQUATIONS

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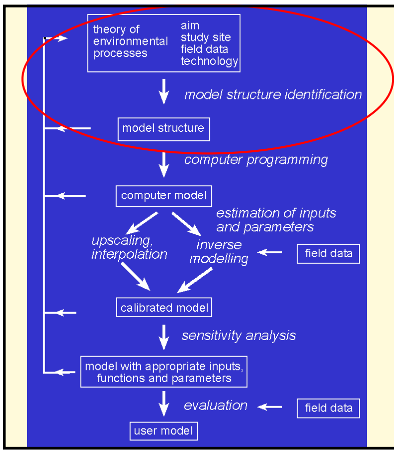

One approach: differential equations

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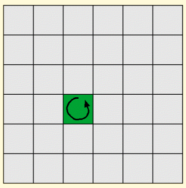
Differential equations



- in dynamic models
  - give rate of change
- mainly point functions
- often used in model structure
  - vegetation growth
  - flow of water into or out of storages
  - radio-active decay
  - etc, etc
- need to be integrated to be used in a model
  - analytical
  - numerical

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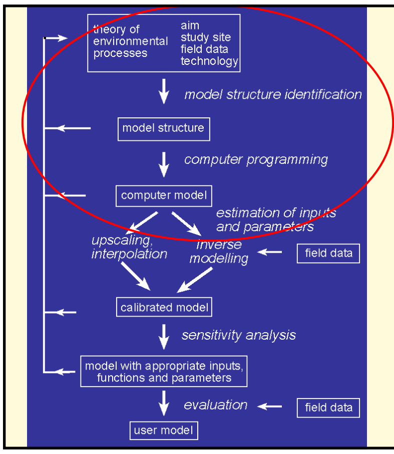

Differential equation

Numerical solution of differential equation

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What is a differential equation?

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Example: interception storage

$$y(t+\Delta t) = y(t) + k \cdot y(t) \cdot \Delta t$$

y	amount of water in interception storage (m)
k	fraction of water in the interception storage that leaves the interception storage per second (s^{-1}, negative value)
Δt	time step length (s)

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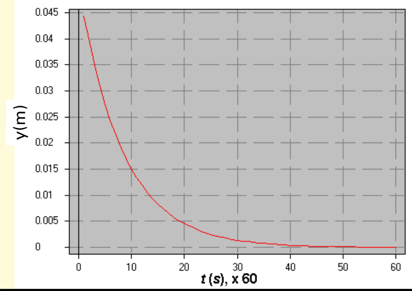
example

Example: interception storage

$$y(t+\Delta t) = y(t) + k \cdot y(t) \cdot \Delta t$$

k = -0.002 s^{-1}
y(0) = 0.05 *m*



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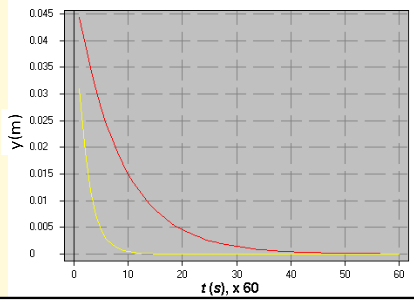
example

Example: interception storage

$$y(t+\Delta t) = y(t) + k \cdot y(t) \cdot \Delta t$$

red line
k = -0.002 s^{-1}
y(0) = 0.05 m

yellow line
k = -0.008 s^{-1}
y(0) = 0.05 m



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Example: interception storage

$$y(t+\Delta t) = y(t) + k \cdot y(t) \cdot \Delta t$$

Can be rewritten:

$$\left.\begin{array}{l} y(t+\Delta t) - y(t) = k \cdot y(t) \cdot \Delta t \\ \Delta y(t) = y(t+\Delta t) - y(t) \end{array}\right\} \quad \Leftrightarrow$$

$$\Delta y(t) = k \cdot y(t) \cdot \Delta t \quad \Leftrightarrow$$

$$\frac{\Delta y(t)}{\Delta t} = k \cdot y(t)$$

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example

Example: interception storage

$$\frac{\Delta y(t)}{\Delta t} = k \cdot y(t)$$

By taking the limit:

$$\frac{dy(t)}{dt} = k \cdot y(t)$$

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example

Example: interception storage

$$\frac{dy(t)}{dt} = k \cdot y(t)$$

is mostly written as:

$$\frac{dy}{dt} = ky$$

This is a differential equation because it involves the derivative

$$\frac{dy}{dt}$$

Of the 'unknown function'

$$y = h(t)$$

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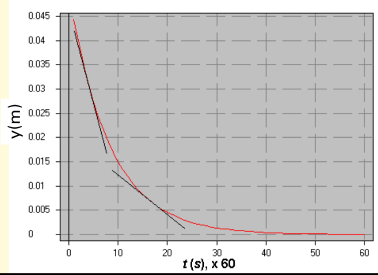
example

Example: interception storage

$$\frac{dy}{dt} = ky$$

Interpretation:
both sides of =
give the 'rate of
change'

Graphical: the slope



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solving

Solving the differential equation

In a model, the differential equation

$$\frac{dy}{dt} = ky$$

Needs to be solved to get a function

$$y = h(t)$$

(in a model, *t* can be filled in for any time step and we get *y*)

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solving

Solving the differential equation

Solving a differential equation can be done in two ways:

- Analytical
- Numerical mathematics

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Analytical solution

Example, analytical solution: initial value problem

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(t_0) = y_i$$

Is (by integration)

$$y(t) = y_i e^{kt}$$

With

y_i initial condition of *y* (at *t*=0), i.e. initial amount of water in interception store

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Analytical solution

Analytical solution

```
import math

k = -0.002             # fraction output from canopy (s-1)
Dt = 60                # timestep (s)

yZero = 0.05
t = 0

while t < 60:
  T = float(t) * Dt
  y = yZero * math.exp(T * k)   # y is not on right side !!
  print(y)
  t = t + 1
```

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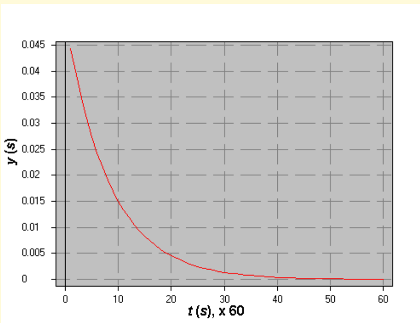
Analytical solution

Analytical solution



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Numerical solution

Often, numerical solutions are used

- Many differential equations cannot be solved analytically
- Numerical solutions are relatively simple to program (not all)
- Numerical solutions are sufficiently precise for most applications
- Modellers can't do maths...

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Many numerical solution algorithms are available

- Euler method
- Heun's method
- Runge-Kutta method

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Euler method or Euler-Cauchy method

The solution of the initial value problem

$$\frac{dy}{dt} = f(y,t), \qquad y(t_0) = c$$

Is (Euler or Euler-Cauchy method):

$$y(t+\Delta t) = y(t) + \Delta t \cdot f(y(t),t)$$

with:
Δt time step length

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Euler method or Euler-Cauchy method

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Is (Euler or Euler-Cauchy method):

$$y(t+\Delta t) = y(t) + \Delta t \cdot f(y(t),t)$$

with:
Δt time step length

Exercise:

$$\frac{dy}{dt} = ky, \qquad y(t_0) = y_i$$

$y(t_0) = 0.02$
$k = -0.002$
Calculate $y(t+60)$
Use a time step of 60

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Euler method or Euler-Cauchy method, example

We have the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(t_0) = y_i$$

The solution is: (Euler or Euler-Cauchy method):

$$y(t+\Delta t) = y(t) + \Delta t \cdot (ky)$$

Note: this is how we solved it initially

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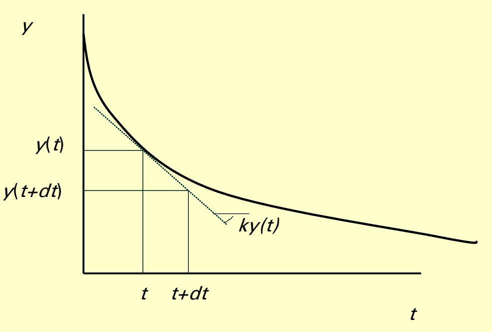

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Euler method or Euler-Cauchy method, example

```
k = -0.002;            # fraction output from canopy (s-1)
Dt = 60;               # timestep (s)

y = 0.05
t = 0

while t < 60
  y = y + Dt * k * y. # y is on the right side!
  print(y)
  t = t + 1
```

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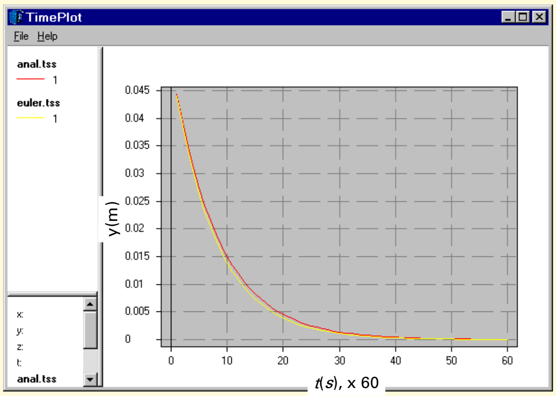
Euler



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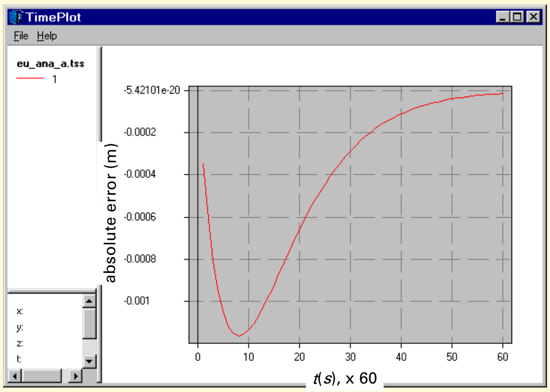
Euler



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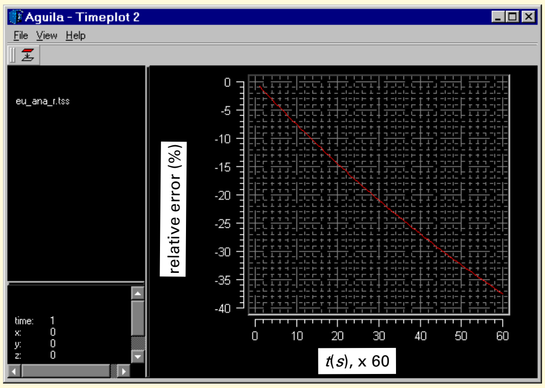
Euler



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Heun

Heun's method

The solution of the initial value problem

$$\frac{dy}{dt} = f(y,t), \qquad y(t_0) = c$$

is:

$$y(t+\Delta t) = y(t) + \Delta t \cdot \frac{f(y(t),t) + f(y^*(t+\Delta t), t+\Delta t)}{2},$$

with:

$$y^*(t+\Delta t) = y(t) + \Delta t \cdot f(y(t),t)$$

(note: $y^*(t+\Delta t)$ is calculated with Euler's method)

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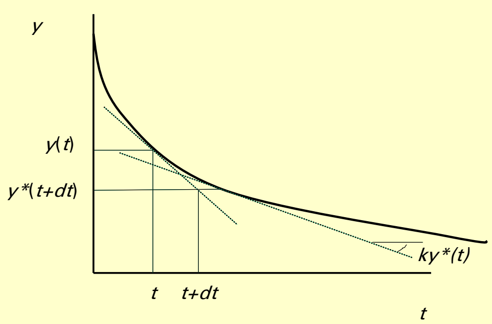

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Heun

Exercise

The solution of the initial value problem

$$\frac{dy}{dt} = f(y,t), \qquad y(t_0) = c$$

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Exercise:

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$y(t_0) = 0.02$
$k = -0.002$
Calculate $y(t+60)$
Use a time step of 60

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Heun's method, example

We have the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(t_0) = y_i$$

The solution is:

$$y(t + \Delta t) = y(t) + \Delta t \cdot \frac{ky + ky^*}{2},$$

with

$$y^*(t + \Delta t) = y(t) + \Delta t \cdot (ky)$$

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Heun's method, example

```
k = -0.002;           # fraction output from canopy (s-1)
Dt = 60;              # timestep (s)

y = 0.05
t = 0

while t < 60
  Ystar = y + Dt * k * y
  y = y + Dt *(( k * y + k * YStar) / 2)
  print(y)
  t = t + 1
```

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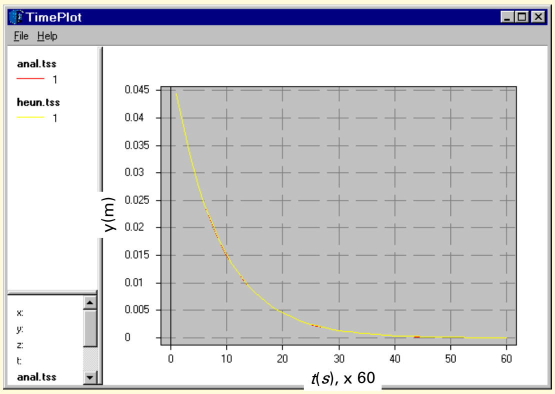

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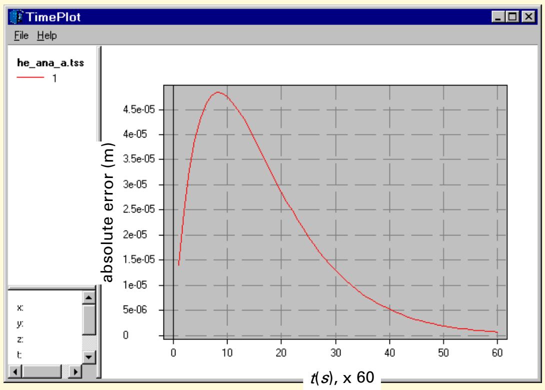

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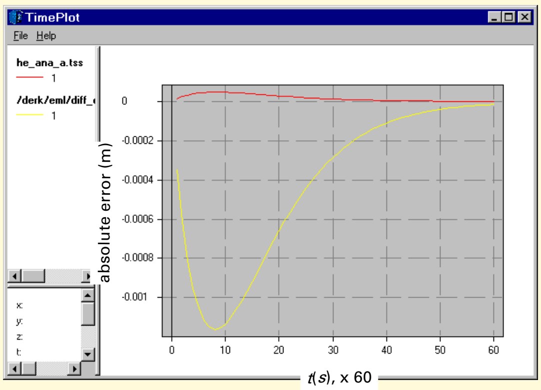

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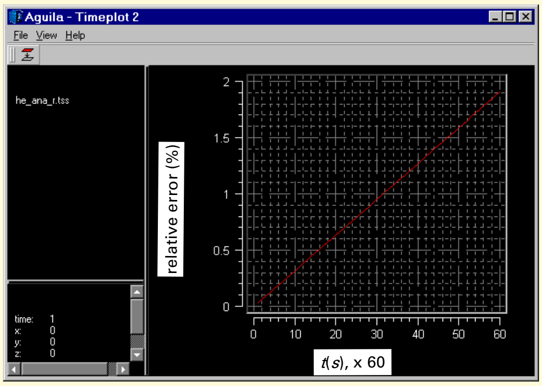

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Classical Runge-Kutta method of 4th order

- calculate four auxilliarry variables

- derive new value from these

- small numerical error, easy to program

Classical Runge-Kutta method of 4th order

We have the initial value problem

$$\frac{dy}{dt} = f(y,t), \qquad y(t_0) = c$$

The solution is

$$k_1 = \Delta t \cdot f\left(y(t),t\right)$$

$$k_2 = \Delta t \cdot f\left(y(t) + \tfrac{1}{2}k_1, t\right)$$

$$k_3 = \Delta t \cdot f\left(y(t) + \tfrac{1}{2}k_2, t\right)$$

$$k_4 = \Delta t \cdot f\left(y(t) + k_3, t\right)$$

$$y(t+\Delta t) = y(t) + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Classical Runge-Kutta method of 4th order

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$$\frac{dy}{dt} = f(y,t), \qquad y(t_0) = c$$

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$$k_1 = \Delta t \cdot f\left(y(t),t\right)$$

$$k_2 = \Delta t \cdot f\left(y(t) + \tfrac{1}{2}k_1, t\right)$$

$$k_3 = \Delta t \cdot f\left(y(t) + \tfrac{1}{2}k_2, t\right)$$

$$k_4 = \Delta t \cdot f\left(y(t) + k_3, t\right)$$

$$y(t+\Delta t) = y(t) + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Exercise:

$$\frac{dy}{dt} = ky, \qquad y(t_0) = y_i$$

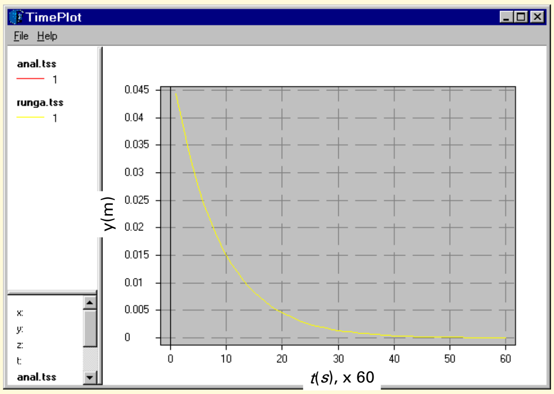
$y(t_0)$ = 0.02
k = -0.002
Calculate $y(t + 60)$
Use a time step of 60

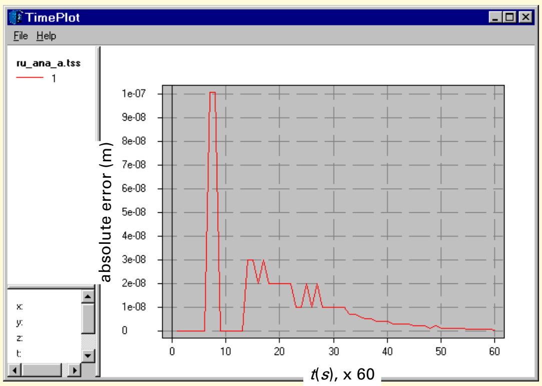
Runge-Kutta method, example

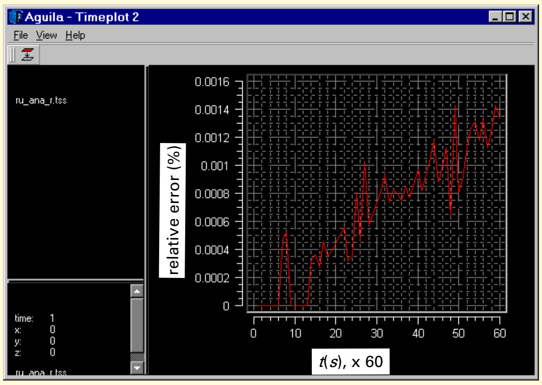
```
k = -0.002;            # fraction output from canopy (s-1)
Dt = 60;               # timestep (s)

y = 0.05
t = 0

while t < 60
  KOne = Dt * (k*y)
  KTwo = Dt * (k*(y + 0.5*KOne))
  KThree = Dt *(k*(y + 0.5*KTwo))
  KFour = Dt *(k*(y + KThree))
  y = y + (1.0/6.0)*(Kone + 2.0 * KTwo + 2.0 * KThree + KFour)
  print(y)
  t = t + 1
```





Some conclusions, final remarks

- in many cases, Euler method can be used (and is used)
- when precision is important, use Runge-Kutta

Not all diff. equations can be solved with Runge-Kutta!
- For complicated problems, use pre-programmed software
  Example: MODFLOW

Literature: Kreyszig

Kreyszig (1999)	*sheets*
h	Δt
x	t
y_{n+1}	$y(t+\Delta t)$

You could read:
- automatic step size selection (p 945)
- proof of local error (p 947)
- error and step size control (p 949)