SPATIO-TEMPORAL SIMULATION MODELLING: POINT MODELS AND DIFFERENTIAL EQUATIONS

01 Introduction

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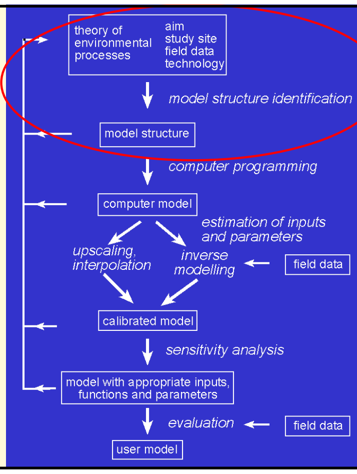

One approach: differential equations

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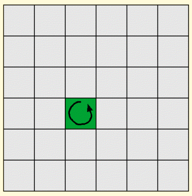
Differential equations

- in dynamic models
  - give rate of change
- mainly point functions
- often used in model structure
  - vegetation growth
  - flow of water into or out of storages
  - radio-active decay
  - etc, etc
- need to be integrated to be used in a model
  - analytical
  - numerical



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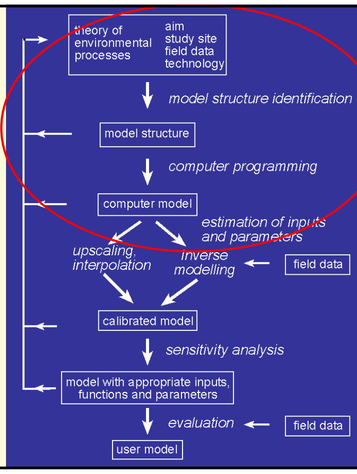

Differential equation

Numerical solution of differential equation

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What is a differential equation?

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Example: interception storage

$$y(t+\Delta t) = y(t) + k \cdot y(t) \cdot \Delta t$$

y	amount of water in interception storage (m)
k	fraction of water in the interception storage that leaves the interception storage per second (s^{-1}, negative value)
Δt	time step length (s)

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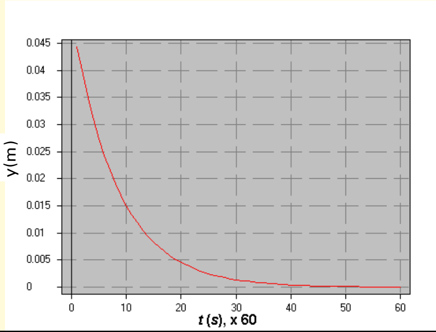
example

Example: interception storage

$$y(t+\Delta t) = y(t) + k \cdot y(t) \cdot \Delta t$$

k = -0.002 s^{-1}
$y(0)$ = 0.05 m



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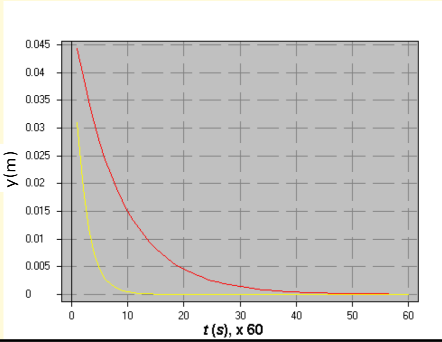
example

Example: interception storage

$$y(t+\Delta t) = y(t) + k \cdot y(t) \cdot \Delta t$$

red line
k = -0.002 s^{-1}
$y(0)$ = 0.05 m

yellow line
k = -0.008 s^{-1}
$y(0)$ = 0.05 m



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example

Example: interception storage

$$y(t+\Delta t) = y(t) + k \cdot y(t) \cdot \Delta t$$

Can be rewritten:

$$\left.\begin{aligned} y(t+\Delta t) - y(t) &= k \cdot y(t) \cdot \Delta t \\ \Delta y(t) &= y(t+\Delta t) - y(t) \end{aligned}\right\} \quad \Leftrightarrow$$

$$\Delta y(t) = k \cdot y(t) \cdot \Delta t \quad \Leftrightarrow$$

$$\frac{\Delta y(t)}{\Delta t} = k \cdot y(t)$$

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example

Example: interception storage

$$\frac{\Delta y(t)}{\Delta t} = k \cdot y(t)$$

By taking the limit:

$$\frac{dy(t)}{dt} = k \cdot y(t)$$

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example

Example: interception storage

$$\frac{dy(t)}{dt} = k \cdot y(t)$$

is mostly written as:

$$\frac{dy}{dt} = ky$$

This is a differential equation because it involves the derivative

$$\frac{dy}{dt}$$

Of the 'unknown function'

$$y = h(t)$$

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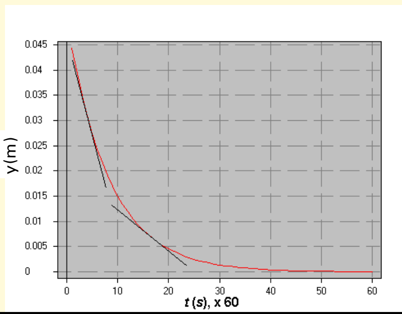
example

Example: interception storage

$$\frac{dy}{dt} = ky$$

Interpretation:
both sides of =
give the 'rate of change'

Graphical: the slope



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solving

Solving the differential equation

In a model, the differential equation

$$\frac{dy}{dt} = ky$$

Needs to be solved to get a function

$$y = h(t)$$

(in a model, *t* can be filled in for any time step and we get *y*)

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solving

Solving the differential equation

Solving a differential equation can be done in two ways:

• Analytical
• Numerical mathematics

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Analytical solution

Example, analytical solution: initial value problem

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(t_0) = y_i$$

Is (by integration)

$$y(t) = y_i e^{kt}$$

With

y_i initial condition of y (at $t = 0$), i.e. initial amount of water in interception store

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Analytical solution

Analytical solution

```
import math

k = -0.002            # fraction output from canopy (s-1)
Dt = 60               # timestep (s)

yZero = 0.05
t = 0

while t < 60:
  T = float(t) * Dt
  y = yZero * math.exp(T * k)  # y is not on right side !!
  print(y)
  t = t + 1
```

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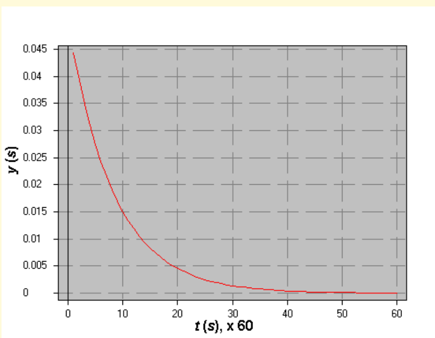
Analytical solution

Analytical solution



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Numerical solution

Often, numerical solutions are used

• Many differential equations cannot be solved analytically

• Numerical solutions are relatively simple to program (not all)

• Numerical solutions are sufficiently precise for most applications

• Modellers can't do maths...

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